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# WILD CANTOR SETS AS APPROXIMATIONS TO CODIMENSION TWO MANIFOLDS

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The main result establishes a technique for constructing wildly embedded Cantor sets in Euclidean $n$-space ($n>2$). For any $(n-2)$-manifold $M$ embedded there with a closed neighborhood $W$ homeomorphic to the product of $M$ and a 2-cell, it gives a Cantor set $C$ in $W$ such that the only loops from the complement of $W$ that are contractible in the complement of $C$ are those that are contractible in the complement of $M$.

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wild embedding
I-essential map
geometrically central
PL product neighborhood
approximable by Cantor sets

## 1. Introduction

Let $C$ denote a compact subset of Euclidean $n$-space, $E^n$. Evidence available for small $n$ (cf. [13, 4]) suggests that in the space $\mathcal{S}$ of all embeddings of $C$ into $E^n$ the pathological, or wild, embeddings are generic; it is widely recognized that, for $n>2$, they are dense in $\mathcal{S}$. Nevertheless, despite their abundance, wild embeddings are seldom quickly or easily described. This paper sets forth a relatively general procedure for constructing pathological embeddings of the Cantor set, the lowest-dimensional object admitting non-standard embeddings in manifolds; with it one can 'approximate' certain PL embedded closed $(n-2)$-manifolds $M$ in $E^n$ by wildly embedded Cantor sets $C$ such that every loop in $E^n-M$ linking $M$ also links $C$.

Bizarre manifestations of Cantor sets have been recognized for most of this century, ever since Antoine's original presentation [1] in 1920 of a strange embedding in $E^3$. Blankinship [2] later extended what Antoine had done to exhibit a strange embedding in $E^n$ for $n>3$. Using techniques of decomposition theory, Bing [3] gave another peculiar embedding in $E^3$; as Lininger [12] and Edwards [9] have pointed out, one can spin Bing's example to determine (also using decomposition-theoretic methods) other peculiar embeddings in $E^n$, $n>3$. Kirkor [10] described a wild Cantor set in $E^3$ having simply connected complement, and more recently DeGryse and Osborne [8] did the same in $E^n$, $n>3$.

All the earlier, pre-existing constructions for $n>3$ are far too uniform to shed any light on the following issue.

**Question.** if $X$ is a compactum in $E^n$ and $U$ is a neighborhood of $X$, is there a Cantor set $C$ in $U$ such that each loop in $E^n-U$ that contracts in $E^n-C$ actually contracts in $E^n-X$?

In terminology introduced by Wright [17], a given compactum $X$ is said to be *approximable by Cantor sets* if the answer to the question above is affirmative for each neighborhood $U$ of $X$. (The question for arbitrary $X$, it should be pointed out, reduces to the one where $X$ is an $(n-2)$-complex in $E^n$.) The dimension of the containing manifold plays a role: for $n=3$ variations to Antoine's methods, detailed in [17], provide an affirmative answer to the question, but for $n>4$ it remains unsolved. Here we prove that each PL $(n-2)$-manifold $X$ whose regular neighborhood in $E^n$ is homeomorphic to the product of $X$ and a 2-cell is approximable by Cantor sets (Theorem 6); moreover, given a closed 1-dimensional subset $L$ of $E^n$, we show how to locate an approximating Cantor set $C$ in $U-L$ (Corollary 10).

One can give a plausible explanation for the additional difficulties encountered when $n>3$. An arbitrary Cantor set $C$ in $E^n$ can be described as the nested intersection of thin regular neighborhoods of $(n-2)$-complexes. Philosophically, exceptional aspects of the 3-dimensional case stem from the fact that $n-2$ happens to be in the trivial range with respect to $n=3$: when $n=3$ one has enough room in $E^n$ for general position adjusting two $(n-2)$-complexes to make them disjoint, but when $n>3$ the comparable adjustment techniques do not have the same effect.

The construction to be set forth here dates back to 1974, when both authors were visitors at the University of Utah, and in the intervening years it has been put to work in several other instances. Daverman exploited the full strength of these techniques in [7], where the construction originally was reported and for which it was designed, to display pathologically embedded $k$-cells in $E^n$ ($n\geq 4$, $n>k\geq 2$) in which each 2-cell is wild. Cannon and Wright [5] extended these methods in producing slippery wild Cantor sets, slippery in the sense that they can be ambiently pushed off any other 0- or 1-dimensional closed subset of $E^n$. Wright [16] put this to work in presenting examples of $n$-dimensional ($n>3$) AR's containing no proper ANR of dimension $>1$, which answered a question of Bing and Borsuk; so did Singh [15] in demonstrating the proliferation of examples like Wright's. Lay also developed an infinite-dimensional analogue to the Cantor set construction in [11].

## 2. I-essential maps and geometrically central sets

The setting for this section consists of three special items: a space $M$ (in practice, a closed PL $n$-manifold $M$), a PL 2-cell $B^2$, and a distinguished point 0 of Int $B^2$. Identify $M$ with the core $M \times \{0\}$ in the product $M \times B^2$.

More generally, suppose $N$ is a PL $(n+2)$-manifold and $M$ is a PL $n$-manifold contained in $N$. Then a *PL product neighborhood $M \times B^2$ of $M$ in $N$* is the image of a PL embedding $e: M \times B^2 \to N$ where $e(\langle x, 0\rangle) = x$ for each $x$ in $M$.

As a pivotal method for measuring the nontriviality of certain fundamental groups, we shall exploit the concept of an I-essential map. Its domain invariably is a disk with holes $H$ (a compact, connected, planar 2-manifold). Given a map $h: H \to M \times B^2$ for which $h(\partial H) \subset M \times \partial B^2$, we say $h$ is *I-inessential* (for: *interior-inessential*) if there exists another map $h': H \to M \times \partial B^2$ with $h'|\partial H = h|\partial H$; when no such map $h'$ exists, we call $h$ *I-essential.*

Additionally, a closed subset $S$ of $M \times B$ is said to be *geometrically central in* $M \times B^2$ if whenever $H$ is a disk with holes and $h: H \to M \times B^2$ is an I-essential map, then $h(H) \cap S \neq \emptyset$. For example, $M$ itself is geometrically central in $M \times B^2$ since $M \times B^2 - M$ retracts to $M \times \partial B^2$.

Nesting characteristics largely account for the applicability of this geometrically central property.

**Proposition 1.** *Let $M$ be a closed, PL n-manifold and let $S_1, S_2, \ldots$ be a (finite or infinite) collection of closed, PL n-manifolds satisfying:*

(1) *each $S_i$ has a PL product neighborhood $S_i \times B^2$ contained in $M \times$* Int $B^2$;

(2) *for $i \geq 1$, $S_{i+1} \times B^2 \subset S_i \times$* Int $B^2$;

(3) *$S_1$ is geometrically central in $M \times B^2$ and, for $i \geq 1$, $S_{i+1}$ is geometrically central in $S_i \times B^2$.*

*Then the set $Y = \bigcap (S_i \times B^2)$ is geometrically central in $M \times B^2$.*

**Proof.** It suffices to show why $S_2$ is geometrically central in $M \times B^2$. To that end, suppose $h: H \to M \times B^2$ is an I-essential map defined on a disk with holes $H$. Adjust $h$ slightly to make its image be in general position with respect to $S_1 \times \partial B^2$, thereby obtaining a new map $g: H \to M \times B^2$ for which

$$g|\partial H = h|\partial H \quad \text{and} \quad g(H) \cap S_2 = h(H) \cap S_2.$$

Consequently, the components of $g^{-1}(S_1 \times B^2)$ are disks with holes, and on at least one such component $\tilde{H}$, $g|\tilde{H}$ is I-essential (for otherwise $S_1$ would fail to be geometrically central). The hypothesis that $S_2$ is geometrically central in $S_1 \times B^2$ implies $g(\tilde{H}) \cap S_2 \neq \emptyset$ and, therefore, $h(H) \cap S_2 \neq \emptyset$.

**Lemma 2.** *Suppose $f: X \to M$ is a map from a metric space $X$ to a manifold $M$; $P$ is a closed, bicollared subset of $M$; $Z = f^{-1}(P)$; $F: Z \times I \to P$ is a homotopy with $F_0 = f|Z$; $U$ is a neighborhood of $F(Z \times I)$ in $M$; and $N$ is a neighborhood of $Z$ in $X$. Then*

*there exist a neighborhood $W$ of $Z$ in $N$ and a map $\tilde{f}: X \to M$ satisfying*:
(1) $\tilde{f}|X - W = f|X - W$;
(2) $\tilde{f}|Z = F_1$; *and*
(3) $\tilde{f}(W - Z) \subset U - P$.

**Proof.** Let $P \times [-1, 1]$ denote a bicollar on $P$ in $M$. Arrange the parameterization so there is a neighborhood $V$ of $F(Z \times I)$ in $P$ with $V \times [-1, 1] \subset U$. Since $V \times [-1, 1]$ is an ANR, by the Borsuk Homotopy Extension Theorem there exist a neighborhood $W$ of $Z$ in $X$ and an extension $G: W \times I \to V \times [-1, 1]$ of $F$ with $G_0 = f|W$. Determine a Urysohn function $u: X \to [0, 1]$ with $u(X - W) = 0$ and $u(Z) = 1$. Letting $p_1: V \times [-1, 1] \to V$ and $p_2: V \times [-1, 1] \to [-1, 1]$ denote the projection maps, define the map $\tilde{f}$ as:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X - W, \\ \langle p_1(G(\langle x, u(x)\rangle)), p_2 f(x)\rangle & \text{if } x \in W. \end{cases}$$

Verifying $\tilde{f}$ has the desired properties is straightforward. □

Next we describe a foundational, 3-dimensional linking apparatus, which will be referred to as a *standard link in* $I \times B^2$. Let $I$ denote the interval $[-1, 1]$. Identify two points $p$ and $q$ in Int $B^2$. Name the straight arc $\alpha$ connecting $p$ and $q$ (view $B^2$ as convex and planar). Let $\alpha_K$, $\alpha_L$ and $\beta$ denote two PL arcs and a PL simple closed curve in $I \times B^2$, linked as indicated in Fig. 1, and let $D_K$, $D_L$ and $D$ be the planar 2-disks bounded by $(\{-1\} \times \alpha) \cup \alpha_K$, $(\{1\} \times \alpha) \cup \alpha_L$, and $\beta$, respectively.

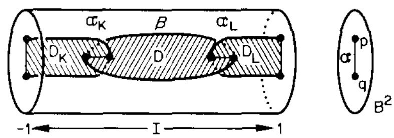


Fig. 1.

Let $K$ be a compact PL $n$-manifold with boundary contained in a closed PL $n$-manifold $M$. The following device describes what will be called the *typical replacement for $M$ in $M \times B^2$*. (See also Fig. 2.) Specify a bicollar $\partial K \times [-1, 1]$ on $\partial K$ with $K \times \{-1\} \subset \text{Int } K$. Define manifolds $K^*$, $T^*$ and $L^*$ in $M \times \text{Int } B^2$ using the standard link $\alpha_K$, $\alpha_L$ and $\beta$ in $I \times B^2$ and setting:

$$K^* = [(K - \partial K \times [-1, 0]) \times \{p, q\}] \cup (\partial K \times \alpha_K),$$

$$T^* = \partial K \times \beta,$$

$$L^* = [(M - (K \cup \partial K \times [0, 1])) \times \{p, q\}] \cup (\partial K \times \alpha_L).$$

Note that $K^* \cong \partial(K \times I)$ and $L^* \cong \partial[(M - \text{Int } K) \times I]$, so each of $K^*$, $T^*$ and $L^*$ is a closed PL $n$-manifold. One can obtain disjoint PL product neighborhoods

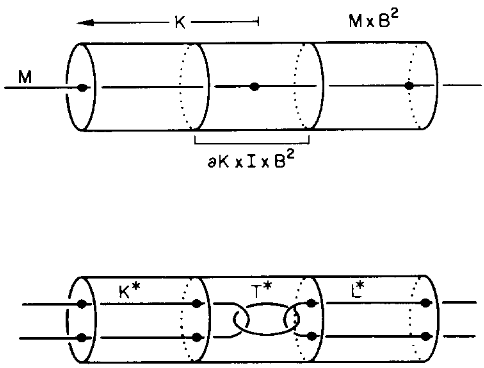


Fig. 2.

$K^* \times B^2$, $T^* \times B^2$ and $L^* \times B^2$ in $M \times \operatorname{Int} B^2$ by thickening the 1-manifolds $\alpha_K$, $\alpha_L$ and $\beta$ in $I \times B^2$ and attaching the product of the result with $\partial K$ to a thickening of the relevant parts of $M \times \{p, q\}$. The typical replacement for $M$ is the manifold $M^* = K^* \cup T^* \cup L^*$.

The crucial feature is spelled out below.

**Proposition 3.** *The typical replacement $M^*$ for $M$ described above is geometrically central in $M \times B^2$.*

**Proof.** Recall the disks $D_K$, $D_L$ and $D$ associated with the standard link in $I \times B^2$. Set

$$K_0 = [(K - \partial K \times [-1, 0]) \times \alpha] \cup (\partial K \times D_K),$$

$$T_0 = \partial K \times D, \quad \text{and}$$

$$L_0 = [(M - (K \cup \partial K \times [0, 1])) \times \alpha] \cup (\partial K \times D_L).$$

Observe that $K_0$ and $L_0$ are 2-sided submanifolds of $M \times \operatorname{Int} B^2$ and that $K^* \cup T_0 \cup L^*$ contains an ambiently equivalent copy of the core $M \times \{0\}$. Thus, $K^* \cup T_0 \cup L^*$ is geometrically central.

Suppose $h : H \to M \times B^2$ is an I-essential map for which $h(H) \cap M^* = \emptyset$. We show this is impossible by establishing the existence of a map $h : H \to M \times B^2$ agreeing with $h$ on $\partial H$ and satisfying

$$h(H) \cap (K^* \cup T_0 \cup L^*) = \emptyset.$$

Alter $h$ slightly so that $h^{-1}(T_0)$ is a finite collection of simple closed curves in Int $H$. It suffices to show how to alter $h$ further so as to remove any one of these simple closed curves, while creating no new intersections with $K^* \cup T_0 \cup L^*$ and while leaving $h$ unchanged on $\partial H$.

At least one component $J$ of $h^{-1}(T_0)$ is innermost in the sense that $H - J$ has a component $E$ for which $J \cup E$ is a disk with holes $H'$ and $h^{-1}(T_0) \cap H' = J$. Let

$P = D - (\partial D \cup \alpha_K \cup \alpha_L)$. Since the loop $h(J)$ is contained in $\partial K \times P$, there is a homotopy $G: J \times I \to \partial K \times P$ such that $G_0 = h$ and $J$ is the union of two arcs $A_1$ and $A_2$ having common endpoints, where $G_1(A_1)$ is a loop in $\partial K \times \mathrm{pt}$ and $G_1(A_2)$ is a loop in $\mathrm{pt} \times P$. Since $\partial K \times P$ is bicollared, we can invoke Lemma 2 to cover the homotopy $G$, exercising suitable controls, thereby obtaining a map $h_1: H \to M \times B^2$ such that $h_1 = h$ outside some small neighborhood $V$ of $J$, $h|J = G|J$, and $h_1^{-1}(K^* \cup T_0 \cup L^*) \cap V = J$. Assume $h_1$ has been chosen so $h_1^{-1}(K_0 \cup L_0) \cap \mathrm{Int}\, H'$ is a finite collection of pairwise disjoint simple closed curves and arcs, all with endpoints in $J$.

Distinguish the sides of $K_0$ and $L_0$ as $K_0^+$, $K_0^-$ and $L_0^+$, $L_0^-$. Orient the arc $A_2$ and associate with each point $z$ of $A_2 \cap h^{-1}(K_0 \cup L_0)$ one of the symbols $K+$ or $K-$ ($L+$ or $L-$) depending on whether the loop $h_1(A_2)$ passes through $K_0$ ($L_0$) from $K_0^+$ to $K_0^-$ ($L_0^+$ to $L_0^-$) or from $K_0^-$ to $K_0^+$ ($L_0^-$ to $L_0^+$) at $h_1(z)$. If two consecutive points of $A_2 \cap h_1^{-1}(K_0 \cup L_0)$ have been assigned 'cancelling' symbols (for example, $K+$ and $K-$), then $h_1$ can be altered, making use of Lemma 2, so as to remove these points of intersection with $K_0 \cup L_0$ while creating no new intersections with $K_0 \cup L_0$. As a consequence, we assume that consecutive points of $A_0 \cap h^{-1}(K_0 \cup L_0)$ do not have cancelling symbols. If, after these adjustments, $h_1(A_2) \cap (K_0 \cup L_0) \neq \emptyset$, then there is an arc $\gamma$ in $H' \cap h_1^{-1}(K_0 \cup L_0)$ innermost in the sense that $\gamma$ together with some component $A_2'$ of $A_2$ forms a simple closed curve in $H'$ with $(\mathrm{Int}_J A_2') \cap h_1^{-1}(K_0 \cup L_0) = \emptyset$. There is a neighborhood $W$ of $\gamma$ in $H'$ such that the component $W'$ of $W - \gamma$ containing points of $\mathrm{Int}_J A_2'$ maps to one particular side of either $K_0$ or $L_0$; for definiteness, say to $K_0^+$. This is a contradiction, because the endpoints of $A_2'$ must then exhibit the cancelling symbols $K+$ and $K-$. We conclude that $h_1(A_2) \cap (K_0 \cup L_0) = \emptyset$.

This being the case, the loop $h_1(A_2)$ is null homotopic in $(\mathrm{pt} \times P) - (K_0 \cup L_0)$. Extend $D$ slightly to a disk $D'$ in $\mathrm{Int}(I \times B^2)$ with $D \subset \mathrm{Int}\, D'$. Name a homotopy $\psi: J \times I \to \partial K \times (D' - (\alpha_K \cup \alpha_L))$ with $\psi_0 = h_1$ and $\psi_1(J) \subset \partial K \times (D' - D)$. Another application of Lemma 2 yields a map $h_2: H \to M \times B^2$ agreeing with $h$ outside some small neighborhood of $J$ and such that $h_2^{-1}(K_0 \cup T_0 \cup L_0) = h^{-1}(K_0 \cup T_0 \cup L_0) - J$. This establishes the reduction discussed earlier and concludes the proof. $\square$

## 3. Constructing wild Cantor sets

There are two key steps in the proposed method of building wild Cantor sets. The first has already been laid out in Proposition 3, detailing the geometric centrality of the typical replacement. The other is presented in the result below, showing how the typical replacement procedure can be applied finitely often to cause a useful metamorphosis, which transforms a given manifold with a PL product neighborhood into *small* pieces, the union of which, of course, is geometrically central and has a PL product neighborhood.

**Theorem 4.** *Let $M$ be a closed, orientable, PL n-manifold and let $\varepsilon > 0$. Then $M \times B^2$ contains a finite collection $R_1, \ldots, R_m$ of closed, orientable, PL n-manifolds having pairwise disjoint PL product neighborhoods $R_1 \times B^2, \ldots, R_m \times B^2$ contained in $M \times B^2$, each of diameter less than $\varepsilon$, and such that $S = R_1 \cup \cdots \cup R_m$ is geometrically central in $M \times B^2$.*

**Proof.** Fix a triangulation $\Sigma$ of $M$ with small mesh. Let $X(i)$ denote the set of all barycenters of $i$-simplexes in $\Sigma$, and let $K(i)$ denote the simplicial neighborhood of $X(i)$ in $\Sigma''$, the second barycentric subdivision of $\Sigma$. The manifolds $K(0), \ldots, K(n)$ cover $M$, have pairwise disjoint interiors and, provided the mesh of $\Sigma$ is small enough, have components of diameter less than $\varepsilon$. Furthermore, for $i > 0$, $K(i)$ meets $K(0) \cup \cdots \cup K(i-1)$ in an $(n-1)$-manifold in $\partial K(i)$. We derive the desired result through $n$ applications of the typical replacement technique.

The initial operation detaches $K(0)$, producing manifolds $K^*(0)$, $T^*(0)$, and $L^*(0)$ having pairwise disjoint PL product neighborhoods $K^*(0) \times B^2$, $T^*(0) \times B^2$, and $L^*(0) \times B^2$ in $M \times \operatorname{Int} B^2$ and with $K^*(0) \cup T^*(0) \cup L^*(0)$ geometrically central in $M \times B^2$. (See Fig. 3.) As noted earlier, $L^*(0)$ may be viewed as the image

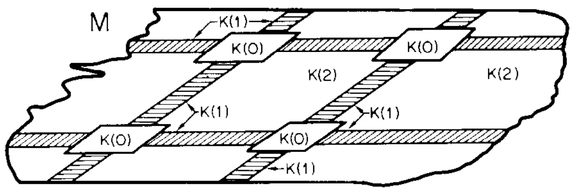


Fig. 3.

of a PL embedding $\phi : \partial((M - \operatorname{Int} K(0)) \times I) \to M \times B^2$. For $i = 1, 2, \ldots, n$ define manifolds $K(i, 1)$ as

$$K(i, 1) = \phi((K(i) \times \partial I) \cup ((\partial K(0) \cap K(i)) \times I)).$$

Impose size controls forcing the components of $K^*(0)$, $T^*(0)$, and also the $K(i, 1)$ to have diameter less than $\varepsilon$. Note that, like the $K(i)$'s in $M$, the manifolds $K(i, 1)$ $(i = 1, \ldots, n)$ cover $L^*(0)$ and have pairwise disjoint interiors.

To iterate, we suppose $L^*(j-1)$ is covered by sets $K(i, j)$ $(i = j, j+1, \ldots, n)$ having components of diameter less than $\varepsilon$ and having pairwise disjoint interiors, and that $L^*(j-1) \times B^2$ already has been identified. The next application of the typical replacement procedure then detaches $K(j, j)$, producing manifolds $K^*(j)$, $T^*(j)$, and $L^*(j)$ analogous to $K^*(0)$, $T^*(0)$, and $L^*(0)$ above, where $L^*(j)$ is covered by manifolds $K(i, j+1)$ $(i = j+1, \ldots, n)$ with pairwise disjoint interiors and small component size. (See Fig. 4.) Note that the $n$th replacement operates on $L^*(n-2) = K(n-1, n-1) \cup K(n, n-1)$, detaching $K(n-1, n-1)$ and leaving only $L^*(n-1)$ with possibly uncontrolled component size. However, the component size

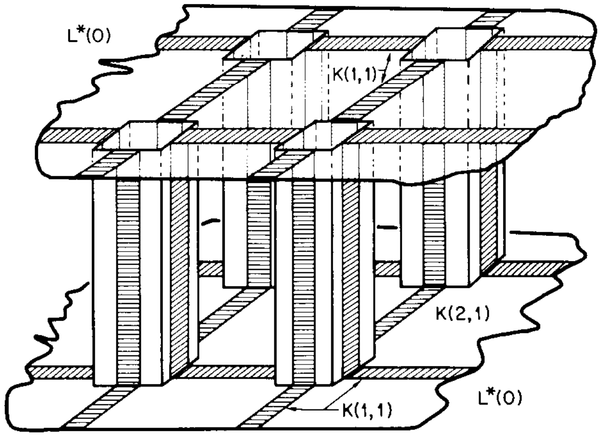


Fig. 4.

does remain small, due to the fact that $L^*(n-1)=K(n,n)$ and to controls on its predecessor $K(n,n-1)$.

The desired manifolds $R_1,\ldots,R_m$ arise as the components of the manifolds $K^*(0),\ldots,K^*(n-1)$, $T^*(0),\ldots,T^*(n-1)$, and $L^*(n-1)$. Proposition 1 and Proposition 3 combine to demonstrate that $R_1\cup\cdots\cup R_m$ is geometrically central in $M\times B^2$. Finally, the orientability of each $R_i$ follows from the orientability of the PL product neighborhood $R_i\times B$, dictated by the orientability of $M\times B^2$. □

Continue to let $M$ denote a closed, orientable, PL $n$-manifold. Now, applying Theorem 4, one obtains a collection $S_1, S_2,\ldots$ of closed $n$-manifolds in $M\times\operatorname{Int} B^2$ such that

(1) $S_1$ has a PL product neighborhood $S_1\times B^2$ in $M\times\operatorname{Int} B^2$ and $S_1$ is geometrically central in $M\times B^2$;

(2) for $i>0$ $S_{i+1}$ has a PL product neighborhood $S_{i+1}\times B^2\subset S_i\times\operatorname{Int} B^2$ and $S_{i+1}$ is geometrically central in $S_i\times B^2$; and

(3) for $i>0$ each component of $S_i\times B^2$ has diameter less than $1/i$.

Proposition 1 attests that $C=\bigcap(S_i\times B^2)$ is geometrically central in $M\times B^2$. Clearly $C$ is 0-dimensional. With extra care in the construction of the $S_i$ we can assure that $C$ is a Cantor set or, alternatively, $C$ must be contained in some Cantor set in $M\times\operatorname{Int} B^2$. Without loss of generality we assume $C$ is a Cantor set.

**Proposition 5.** *Let $M$, $S_i$ $(i=1,2,\ldots)$, and $C$ be as described above, and let $f: D\to M\times B^2$ be a map of a 2-disk $D$ such that $f|\partial D:\partial D\to M\times\partial B^2$ is not homotopically trivial. Then*

$$f(D)\cap C\neq\emptyset.$$

**Proof.** Since $f|\partial D$ is not homotopically trivial in $M\times\partial B^2$, $f$ is 1-essential. The result follows directly from the fact that $C$ is geometrically central in $M\times B^2$. □

Virtually the same kind of argument establishes the following summary result.

**Theorem 6.** *Let $N$ be a closed orientable PL $(n+2)$-manifold and let $M$ be a closed, PL $n$-manifold embedded in $N$ with a PL product neighborhood. Then $M$ is approximable by Cantor sets.*

**Proposition 7.** *Suppose $N$ is a closed orientable PL $(n+2)$-manifold and $A$ is a closed subset of $N$ that admits an essential map $q: A \to S^1$. Then there exists a neighborhood $U$ of $A$, an extension $\hat{q}: \bar{U} \to S^1$ of $q$, and a closed orientable PL $n$-manifold $M$ having a PL product neighborhood $M \times B^2 \subset N - U$ such that whenever $f: D \to N$ is a map of a 2-disk $D$ for which $f(\partial D) \subset U$ and $qf|\partial D: \partial D \to S^1$ is essential, then*

$$f(D) \cap M \neq \emptyset.$$

**Proof.** The neighborhood $U$ and the extension $\hat{q}$ of $q$ to $U$ come about because $S^1$ is an ANR. Assume $\hat{q}$ is defined on Cl $U$. Express $S^1$ as the union of two arcs $A+$ and $A-$ with disjoint interiors and common endpoints $a$ and $b$.

Identify a closed PL $(n+1)$-manifold $M^*$ which is bicollared in $N$ and which separates $\hat{q}^{-1}(a)$ from $\hat{q}^{-1}(b)$ there. Then $M^* \cap \hat{q}^{-1}(A+)$ and $M^* \cap \hat{q}^{-1}(A-)$ are disjoint closed subsets of $M^*$. Choose $M$ to be a closed PL $n$-manifold in $M^*$, bicollared in $M^*$, and separating $M^* \cap \hat{q}^{-1}(A+)$ from $M^* \cap \hat{q}^{-1}(A-)$ in $M^*$.

Clearly $M \subset N -$ Cl $U$. the PL product neighborhood $M \times B$ arises from the two bicollars.

Consider a map $f: D \to N \times B^2$ for which $f(\partial D) \subset U$ and $\hat{q} \cdot f|\partial D$ is essential. Assume without loss of generality that $f^{-1}(M^*)$ is a finite collection of arcs and simple closed curves in $D$. The separating properties of $M^*$ and $M$ plus a simple counting argument lead to an arc $\gamma$ in $f^{-1}(M^*)$ spanning $D$ and revealing the desired conclusion; $f(D) \cap M \neq \emptyset$ because $f(\gamma) \cap M \neq \emptyset$.

**Corollary 8.** *Let $N$ and $A$ be as in Proposition 7. Then there exists a Cantor set $C$ in $N - A$ such that each neighborhood of $A$ contains a loop which cannot be contracted in $N - A$.*

**Proof.** Name $M$, $U$, and $\hat{q}: U \to S^1$ as in Proposition 7. Let $C$ be a Cantor set in $M \times B^2$ as described in Proposition 5. Suppose $f: \partial D \to U$ describes a loop in $U$ for which $\hat{q} \cdot f$ is essential. A contraction of $f(\partial D)$ in $N$ can be viewed as the image $F(D)$ of an extension $F$ of $f$ to all of $D$. Any such contraction $F$ can be modified slightly so that $F|\partial D = f$ and the components of $F^{-1}(M \times B^2)$ are disks with holes. Theorem 6 attests $F(D) \cap M \neq \emptyset$, for each such contraction $F$, so $F|H: H \to M \times B^2$ must be I-essential on some component $H$ of $F^{-1}(M \times B^2)$. Proposition 1 and the definition of $C$ then certify that $F(D) \cap C \neq \emptyset$.

What remains to be seen is why each neighborhood of $A$ contains such a loop $f(\partial D)$. Consider any neighborhood $V$ of $A$ and assume $V \subset U$ so that $\hat{q}|V: V \to S^1$

is essential. Obtain an open cover $\mathcal{W}$ of $A$ by pathwise connected subsets of $V$, and let $\mathcal{N}$ denote the nerve of this cover. Find maps $c: A \to \mathcal{N}$ and $d: \mathcal{N} \to S^1$ such that $d \cdot c$ is homotopic to $\hat{q}|A$. Thus, $d$ must be essential. Its restriction to the 1-skeleton $\mathcal{N}^{(1)}$ of $\mathcal{N}$ can be arranged to factor as $\hat{q} \cdot e$, where $e: \mathcal{N}^{(1)} \to V$. Since $S^1$ is aspherical, $\hat{q} \cdot e: \mathcal{N}^{(1)} \to S^1$ is essential. Hence, one can build a map $\eta: S_1 \to \mathcal{N}^{(1)}$ such that $\hat{q} \cdot e \cdot \eta$ is essential, and $f = e \cdot \eta$ is the required map. □

## 4. Dodging a 1-dimensional set

A technical improvement to Theorem 6, which pinpoints more precisely the possible locations of the wild Cantor set $C$ approximating the manifold $M$, is presented in this section. Similar results are derived in [5] and [7].

**Theorem 9.** *Let $M$ be a closed, orientable, PL $n$-manifold, and let $L$ be a closed 1-dimensional subset of $M \times B^2$. Then there exists a closed, orientable $n$-manifold $P$ such that $P$ is geometrically central in $M \times B^2$ and has a PL product neighborhood in $(M \times \text{Int } B^2) - L$.*

**Proof.** Since $L$ is 1-dimensional, the origin $0 \in B^2$ is an unstable value of $p|L$, where $p$ denotes the projection $M \times B^2 \to B^2$. Hence, $p$ can be adjusted, rel $M \times \partial B^2$, to a PL map $p'$ such that $p'(L)$ misses a neighborhood of 0. Let $F: M \times B^2 \times I \to B^2$ denote this homotopy, with $F_0 = p$ and $F_1 = p'$. Standard techniques from PL topology allow us to assume $F$ is simplicial with respect to triangulations $\Sigma$ of $M \times B^2 \times I$ and $T$ of $B^2$, and with respect to the first barycentric subdivisions $\Sigma'$ and $T'$ of $\Sigma$ and $T$. Furthermore, we assume that 0 is a vertex of some simplex in $T'$ and that $F_1(L) \cap \text{st}(0, T') = \emptyset$. Cohen's transversality theorem [6] shows that $Q = F^{-1}(0)$ is a compact PL$(n+1)$-manifold (with boundary) with a PL product neighborhood and that $\partial Q = P \cup (M \times \{0\})$, where $P = F_1^{-1}(0)$. The PL product neighborhood $P \times B^2$ can be obtained in canonical fashion as $F_1^{-1}(\text{st}(0, T''))$, where $T''$ is the second barycentric subdivision of $T$. The only feature left to be explained is why $P$ is geometrically central in $M \times B^2$.

Suppose there exists an I-essential map $h: H \to M \times B^2$ of a disk with holes $H$ for which $h(H) \cap P = \emptyset$. We will attain a contradiction by describing how $h$ can be redefined to avoid $M$, which from here on we identify with $M \times \{0\} \subset M \times B^2$.

Adjust $h$ slightly, rel $\partial H$, to make $h^{-1}(M)$ be a finite set (necessarily nonempty) of points, and let $h^*: H \times I \to M \times B^2 \times I$ be defined as $h^* = h \times \text{Id}$. Take a general position approximation, affecting no points of $(\partial H \times I) \cup (H \times \{0, 1\})$, so that $h^{*-1}(Q)$ is a finite, pairwise disjoint collection of simple closed curves and arcs, each embedded by $h^*$. Since $h(H) \cap M \neq \emptyset$, at least one arc, say $\alpha$, in this collection has endpoints $\langle y, 0\rangle$ and $\langle z, 0\rangle$. We achieve our goal by demonstrating how to modify $h$, rel $\partial H$, so as to remove the points $h(y)$ and $h(z)$ from $M$ while creating no new intersections with $M$. Let $D_y$ and $D_z$ be disjoint 2-disk neighborhoods of $y$ and $z$

in $H$ such that there is a 2-disk $D$ in $B^2$ with $p \cdot h \mid D_y : D_y \to D$ and $p \cdot h \mid D_z : D_z \to D$ being homeomorphisms. Without loss of generality,

$$(Q \times B^2) \cap (M \times B^2 \times \{0\}) = M \times D.$$

Name homeomorphisms $e_y$ and $e_z$ of $S^1$ onto $\partial D_y$ and $\partial D_z$, respectively, with $p \cdot h \cdot e_y = p \cdot h \cdot e_z$. The component $T$ of $(ph^*)^{-1}(D)$ containing $\alpha$ is topologically equivalent to $I \times B^2$ via a homeomorphism $\psi$ sending $\{0\} \times B^2$ onto $D_y$ and $\{1\} \times B^2$ onto $D_z$, and $\psi$ can be defined so that, for $x \in S^1$, $\psi(\{0\} \times \{x\}) = e_y^{-1}(x)$. Consequently, $p \cdot h^* \mid \psi(I \times S^1)$ induces a homotopy $G : S^1 \times I \to \partial D$ with $G_0 = p \cdot h \cdot e_y$, and clearly then $G_1$ is homotopic to $p \cdot h \cdot e_z$. Finally, note that the presence of $T$ in $H \times I$ shows $H$ has an orientation whose restriction to $D_y$ and $D_z$ is compatible with those transferred by $e_y$, $e_z$, respectively, on their boundaries.

Let $A$ be a PL arc in Int $H$ joining $D_y$ and $D_z$ and for which $h(A) \cap M = \emptyset$. The discussion in the preceding paragraph shows the loop

$$(h \mid \partial D_y) \cdot (h \mid A) \cdot (h \mid \partial D_z)^{-1} \cdot (h \mid A)^{-1}$$

to be trivial in $(M \times B^2) - M$. Adjust $h$ near $D_y \cup A \cup D_z$ and obtain a 2-cell $D^*$ containing $D_y \cup A \cup D_z$ such that $p \cdot h \mid \partial D^*$ is the trivial loop named above (this step also depends on the orientation compatibility observation of the preceding paragraph). Then $h \mid \partial D^*$ extends over $D^*$ to a map $h_1 : D^* \to (M \times B^2) - M$. This removes the intersections $h(y)$ and $h(z)$ with $M$, as required. □

As in the preceding section, this yields the following corollary.

**Corollary 10.** *Suppose $N$ is a closed, orientable, PL$(n+2)$-manifold, $M$ is a closed $n$-manifold embedded in $N$ with a PL product neighborhood $W$, and $L$ is a closed 1-dimensional subset of $N$. Then there exists a Cantor set $C$ in $W - L$ such that $C$ is geometrically central in $W$. Furthermore, each loop in $N - W$ that contracts in $N - C$ also contracts in $N - M$.*

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